NUMERICAL METHODS FOR SYSTEMS OF CONSERVATION LAWS AND RELATED EQUATIONS WITH APPLICATIONS IN MINERAL PROCSSING*

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Abstract. This contribution provides a survey on the mathematical and numerical treatment of first-order systems of conservation of conservation laws with applications to models related to mineral processing. In one space dimension, the governing equations can be written as

$$\partial_t \Phi(x,t) + \partial_x \boldsymbol{f}\big(\Phi(x,t)\big) = \mathbf{0},\ \Phi = (\phi_1,\dots,\phi_N)^{\mathrm{T}},\ \boldsymbol{f}(\Phi) = \big(f_1(\phi),\dots,f_N(\Phi)\big)^{\mathrm{T}},\ x \in \mathbb{R},\ t > 0, \quad (1)$$

where t is time, x is spatial position, $\Phi = \Phi(x,t)$ is the vector of N scalar unknowns that in many contexts represent densities (concentrations or volume fractions), $\boldsymbol{f} = \boldsymbol{f}(\Phi)$ is the given flux density vector, and (1) is solved together with initial or initial-boundary conditions. For $N = 1$, equation (1) reduces to the scalar conservation law

$$\partial_t \phi(x,t) + \partial_x f\big(\phi(x,t)\big) = 0, \quad x \in \mathbb{R}, \quad t > 0. \quad (2)$$

In a first part we recall some basic facts about the system (1), recalling the concepts of conservation, hyperbolicity, the method of characteristics, jump conditions, the Riemann problem, and entropy conditions. The most important property that affects the mathematical and numerical analysis of (1) or (2) is the fact that solutions of these equations become, in general, discontinuous after a finite time, even when the initial data are smooth, and need to be defined in a particular weak sense. The discontinuities formed by solutions of (systems of) nonlinear conservation laws are known as shocks.

In a second part we address schemes suitable for the numerical approximation of solutions of (1) or (2). The simplest methods are explicit conservative finite difference schemes [25–30]. It is known that monotone schemes, such as the Lax-Friedrichs, Godunov, or Engquist-Osher schemes, provably converge to an entropy solution of the initial value problem for (2). However, these schemes are only first-order accurate, and one wishes to design (more efficient) schemes with higher-order accuracy. One popular way to achieve this are weighted essentially non-oscillatory (WENO) reconstructions [7]. The second part concludes with a survey of current research problems that are typical of applications in mineral processing, such as conservation laws with discontinuous flux [8] and singular source terms and degenerate parabolic partial differential equations.

Finally, some examples of applications to mineral processing are reviewed. These include the sedimentation of suspensions of small solid particles in a fluid [9–11], polydisperse suspensions [12–15], flocculated suspensions forming compressible sediments [13, 16–18], continuous sedimentation in clarifier-thickeners [17, 19, 20], flotation columns [21–23], and settling in inclined vessels [24].

References

[1] R.J. Le Veque, *Numerical Methods for Conservation Laws*, 2nd ed., Birkhäuser Verlag, Basel 1992.
[2] D. Kröner, *Numerical Schemes for Conservation Laws*, Wiley-Teubner, Chichester & Stuttgart, 1997.

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[3] R.J. Le Veque, *Finite Volume Methods for Hyperbolic Problems*, Cambridge University Press, 2002.
[4] R.J. Le Veque, *Finite Difference Methods for Ordinary and Partial Differential Equations: Steady State and Time Dependent Problems*, SIAM, Philadelphia, 2007.
[5] E.F. Toro, *Riemann Solvers and Numerical Methods for Fluid Dynamics. A Practical Introduction*, Third Ed., Springer-Verlag, Berlin, 2009.
[6] H. Holden & N.H. Risebro, *Front Tracking for Hyperbolic Conservation Laws*, Second Edition, Springer, Berlin, 2015.
[7] C.-W. Shu, 'Essentially Non-Oscillatory and Weighted Essentially Non-Oscillatory Schemes for Hyperbolic Conservation Laws', in A. Quarteroni (Ed.), *Advanced Numerical Approximation of Nonlinear Hyperbolic Equations*, Lecture Notes in Mathematics vol. 1697, Springer-Verlag, Berlin (1998), 325–432.
[8] R. Bürger & K.H. Karlsen, 'Conservation laws with discontinuous flux: a short introduction', *J. Eng. Math.* **60** (2008), 241–247.
[9] G.J. Kynch, 'A theory of sedimentation', *Trans. Faraday Soc.* **48** (1952), 166–176.
[10] R. Bürger, J. Careaga & S. Diehl, 'Entropy solutions of a scalar conservation law modeling sedimentation in vessels with varying cross-sectional area', *SIAM J. Appl. Math.* **77** (2017), 789–811.
[11] R. Bürger, J. Careaga & S. Diehl, 'Flux identification of scalar conservation laws from sedimentation in a cone', *IMA J. Appl. Math.* **83** (2018), 526–552.
[12] R. Bürger, K.H. Karlsen, E.M. Tory & W.L. Wendland, 'Model equations and instability regions for the sedimentation of polydisperse suspensions of spheres', *ZAMM Z. Angew. Math. Mech.* **82** (2002), 699–722.
[13] S. Berres, R. Bürger, K.H. Karlsen & E.M. Tory, 'Strongly degenerate parabolic-hyperbolic systems modeling polydisperse sedimentation with compression', *SIAM J. Appl. Math.* **64** (2003), 41–80.
[14] S. Berres, R. Bürger and E.M. Tory, 'On mathematical models and numerical simulation of the fluidization of polydisperse suspensions', *Appl. Math. Modelling* **29** (2005), 159–193.
[15] R. Bürger, R. Donat, P. Mulet & C.A. Vega, 'Hyperbolicity analysis of polydisperse sedimentation models via a secular equation for the flux Jacobian', *SIAM J. Appl. Math.* **70** (2010), 2186–2213.
[16] M.C. Bustos, F. Concha, R. Bürger & E.M. Tory, *Sedimentation and Thickening: Phenomenological Foundation and Mathematical Theory*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1999.
[17] R. Bürger, K.H. Karlsen and J.D. Towers, 'A model of continuous sedimentation of flocculated suspensions in clarifier-thickener units', *SIAM J. Appl. Math.* **65** (2005), 882–940.
[18] R. Bürger, J. Careaga, S. Diehl and R. Pineda, 'Numerical schemes for a moving-boundary convection-diffusion-reaction model of sequencing batch reactors', *ESAIM: Math. Model. Numer. Anal.* **57** (2023), 2931–2976.
[19] R. Bürger, K.H. Karlsen, C. Klingenberg & N.H. Risebro, 'A front tracking approach to a model of continuous sedimentation in ideal clarifier-thickener units', *Nonlin. Anal. Real World Appl.* **4** (2003), 457–481.
[20] R. Bürger, K.H. Karlsen, N.H. Risebro & J.D. Towers, 'Well-posedness in BV_t and convergence of a difference scheme for continuous sedimentation in ideal clarifier-thickener units', *Numer. Math.* **97** (2004), 25–65.
[21] R. Bürger, S. Diehl & M.C. Martí, 'A system of conservation laws with discontinuous flux modelling flotation with sedimentation', *IMA J. Appl. Math.* **84** (2019), 930–973.
[22] R. Bürger, S. Diehl, M.C. Martí and Y. Vásquez, 'A degenerating convection-diffusion system modelling froth flotation with drainage', *IMA J. Appl. Math.* **87** (2022), 1051–1090.
[23] R. Bürger, S. Diehl, M.C. Martí and Y. Vásquez, 'A difference scheme for a triangular system of conservation laws with discontinuous flux modeling three-phase flows', *Netw. Heterog. Media* **18** (2023), 140–190.
[24] F. Betancourt, R. Bürger, J. Careaga & L. Romero, 'Coupled finite volume methods for settling in inclined vessels with natural convection'. Preprint 2024-09, Centro de Investigación en Ingeniería Matemática, Universidad de Concepción; submitted.
[25] H. Holden & N.H. Risebro, *Front Tracking for Hyperbolic Conservation Laws*, Second Edition, Springer, Berlin, 2015.
[26] D. Kröner, *Numerical Schemes for Conservation Laws*, Wiley-Teubner, Chichester & Stuttgart, 1997.
[27] R.J. Le Veque, *Numerical Methods for Conservation Laws*, 2nd ed., Birkhäuser Verlag, Basel 1992.
[28] R.J. Le Veque, *Finite Volume Methods for Hyperbolic Problems*, Cambridge University Press, 2002.
[29] R.J. Le Veque, *Finite Difference Methods for Ordinary and Partial Differential Equations: Steady State and Time Dependent Problems*, SIAM, Philadelphia, 2007.
[30] E.F. Toro, *Riemann Solvers and Numerical Methods for Fluid Dynamics. A Practical Introduction*, Third Ed., Springer-Verlag, Berlin, 2009.